

# MULTI-FREQUENCY CALDERÓN-ZYGMUND ANALYSIS AND CONNEXION TO BOCHNER-RIESZ MULTIPLIERS

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**ABSTRACT.** In this work, we describe several results exhibited during a talk at the *El Escorial 2012* conference. We aim to pursue the development of a multi-frequency Calderón-Zygmund analysis introduced in [10]. We set a definition of general multi-frequency Calderón-Zygmund operators. Unweighted estimates are obtained using the corresponding multi-frequency decomposition of [10]. Involving a new kind of maximal sharp function, weighted estimates are obtained.

The so-called Calderón-Zygmund theory and its ramifications have proved to be a powerful tool in many aspects of harmonic analysis and partial differential equations. The main thrust of the theory is provided by

- the Calderón-Zygmund decomposition, whose impact is deep and far-reaching. This decomposition is a crucial tool to obtain weak type  $(1, 1)$  estimates and consequently  $L^p$  bounds for a variety of operators;
- the use of the “local” oscillation  $f - (f_Q f)$  (for  $Q$  a ball). These oscillations appear in the elementary functions of the “bad part” coming from the Calderón-Zygmund decomposition and in the definition of the maximal sharp function, which allows to get weighted estimates.

The oscillation  $f - (f_Q f)$  can be seen as the distance between the function  $f$  and the set of constant functions on the ball  $Q$ , indeed the average is the best way to locally approximate the function by a constant. By this way, the constant function being associated to the frequency 0, we understand how the classical Calderón-Zygmund theory is related to the frequency 0.

As for example, well-known Calderón-Zygmund operators are the Fourier multipliers associated to a symbol  $m$  satisfying Hörmander’s condition

$$|\partial^\alpha m(\xi)| \lesssim |\xi|^{-|\alpha|} = d(\xi, 0)^{-|\alpha|},$$

which encodes regularity assumption of the symbol relatively to the frequency 0.

In this work, we are interested in the extension of this theory with respect to a collection of frequencies and we focus on sharp constants relatively to the number of the considered frequencies.

Such questions naturally arise as soon as we work on a multi-frequency problem:

- Uniform bounds for a Walsh model of the bilinear Hilbert transform (see [12] by Oberlin and Thiele);

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- A variation norm variant of Carleson's theorem (see [11] by Oberlin, Seeger, Tao, Thiele and Wright);
- Such a multi-frequency Calderón-Zygmund was introduced by Nazarov, Oberlin and Thiele in [10] for proving a variation norm variant of a Bourgain's maximal inequality.

Similarly to the fact that a Fourier multiplier with a symbol satisfying Hörmander's condition is a classical Calderón-Zygmund, we may extend this property to a collection of frequencies. More precisely, let  $\Theta := (\xi_1, \dots, \xi_N)$  be a collection of frequencies and consider a symbol  $m$  verifying for all multi-indices  $\alpha$

$$|\partial^\alpha m(\xi)| \lesssim d(\xi, \Theta)^{-|\alpha|},$$

with  $d(\xi, \Theta) := \min_{1 \leq i \leq N} |\xi - \xi_i|$ . Such symbols give rise to Fourier multipliers, which should be the prototype of what we want to call *multi-frequency Calderón-Zygmund operators*.

In the 1-dimensional setting with a collection of frequencies  $\Theta := (\xi_1, \dots, \xi_N)$  (assumed to be indexed by the increasing order  $\xi_1 < \xi_2 < \dots < \xi_N$ ), an example is given by the multi-frequency Hilbert transform which corresponds to the symbol

$$m(\xi) = \begin{cases} -1, & \xi < \xi_1 \\ (-1)^{j+1}, & \xi_j < \xi < \xi_{j+1} \\ (-1)^{N+1}, & \xi > \xi_N. \end{cases}$$

Let us now detail a definition of “multi-frequency Calderón-Zygmund” operator:

*Definition 0.1.* Let  $\Theta := (\xi_1, \dots, \xi_N)$  be a collection of  $N$  frequencies of  $\mathbb{R}^n$ . An  $L^2$ -bounded linear operator  $T$  is said to be a Calderón-Zygmund operator relatively to  $\Theta$  if there exist operators  $(T_j)_{j=1, \dots, N}$  and kernels  $(K_j)_{j=1, \dots, N}$  verifying

- Decomposition:  $T = \sum_{j=1}^N T_j$ ;
- Integral representation of  $T_j$ : for every function  $f \in L^2$  compactly supported and  $x \in \text{supp}(f)^c$ ,

$$T_j(f)(x) = \int K_j(x, y) f(y);$$

- Regularity of the modulated kernels: for every  $x \neq y$

$$\sum_{j=1}^N \left| \nabla_{(x, y)} e^{i\xi_j \cdot (x-y)} K_j(x, y) \right| \lesssim |x - y|^{-n-1}.$$

*Remark 0.1.* As usual, we can weaken the regularity assumption and just require an  $\epsilon$ -Hölder regularity on the modulated kernels.

*Remark 0.2.* If the decomposition is assumed to be orthogonal (which means that for  $i \neq j$ ,  $T_i T_j^* = 0$ ) then it follows that each operator  $T_j$  is a modulated Calderón-Zygmund operator. Such a multi-frequency Calderón-Zygmund operator can also be pointwisely bounded by a sum of  $N$  modulated (classical) Calderón-Zygmund operators and have the same boundedness properties with an implicit constant of order  $N$ . The aim is to study how this order can be improved using sharp estimates.

We first obtain unweighted estimates for such operators:

**Theorem 0.1.** *Let  $\Theta$  be a collection of  $N$  frequencies and  $T$  an associated multi-frequency Calderón-Zygmund operator. Then*

- for  $p \in (1, \infty)$ ,  $T$  is bounded on  $L^p$  with

$$\|T\|_{L^p \rightarrow L^p} \lesssim N^{\left|\frac{1}{p} - \frac{1}{2}\right|}.$$

- for  $p = 1$ ,  $T$  is of weak-type  $(1, 1)$  with

$$\|T\|_{L^1 \rightarrow L^{1,\infty}} \lesssim N^{\frac{1}{2}}.$$

This theorem relies on an adapted Calderón-Zygmund decomposition introduced in [10] by Nazarov, Oberlin and Thiele. We point out that there the constant  $N^{\frac{1}{2}}$  is shown to be optimal and this is the same for the previous weak-type estimate.

Concerning weighted estimates, it is well-known that linear Calderón-Zygmund operators are bounded on  $L^p(\omega)$  for  $p \in (1, \infty)$  and every weight  $\omega$  belonging to the Muckenhoupt's class  $\mathbb{A}_p$  (see Definitions 1.1 and 1.2 for more details about Muckenhoupt's class  $\mathbb{A}_p$  and Reverse Hölder class  $RH_s$ ). Similar properties are satisfied by the Hardy-Littlewood maximal operator and some other linear operators as Bochner-Riesz multipliers [15, 4] or non-integral operators (like Riesz transforms) [1]. All these boundedness, obtained by using suitable Fefferman-Stein inequalities related to maximal sharp functions, involve weights belonging to the class  $\mathcal{W}^p(p_0, q_0) := \mathbb{A}_{\frac{p}{p_0}} \cap RH_{(\frac{q_0}{p})'}$  for some exponents  $p_0 < q_0$ .<sup>1</sup>

As a consequence, it seems that these classes of weights are well-adapted for proving boundedness of linear operators. Following this observation, we will consider a multi-frequency maximal sharp function, in order to prove weighted estimates for our multi-frequency operators:

**Theorem 0.2.** *Let  $\Theta$  be a collection of  $N$  frequencies. For  $p \in (1, \infty)$ ,  $s \in (1, p)$  and  $t \in (1, \infty)$ , then every multi-frequency Calderón-Zygmund operator  $T$  is bounded on  $L^p(\omega)$  for every weight  $\omega \in RH_{t'} \cap \mathbb{A}_{\frac{p}{s}}$  with*

$$\|T\|_{L^p(\omega) \rightarrow L^p(\omega)} \lesssim N^\gamma$$

and

$$\gamma := \frac{tp}{s \min\{2, s\}} + \left| \frac{1}{2} - \frac{1}{s} \right|.$$

We emphasize that this result is only interesting when  $\gamma < 1$ .

The current paper is organized as follows: after some preliminaries about weights, examples of multi-frequency operators and the main lemma for the multi-frequency analysis, Theorem 0.1 is proved in Section 2. Then in Section 3, we develop the general approach for weighted estimates, based on a suitable maximal sharp function. In Section 4, we describe how this point of view could be used to Bochner-Riesz multipliers.

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<sup>1</sup>From [8], we know that for  $r, s > 1$ ,

$$\mathbb{A}_r \cap RH_s = \{\omega, \omega^s \in \mathbb{A}_{1+s(r-1)}\},$$

so these classes of weights are equivalent to a class of powers of Muckenhoupt's weights.

## 1. NOTATIONS AND PRELIMINARIES

Let us consider the Euclidean space  $\mathbb{R}^n$  equipped with the Lebesgue measure  $dx$  and its Euclidean distance  $|x - y|$ . Given a ball  $Q \subset \mathbb{R}^n$  we denote its center by  $c(Q)$  and its radius by  $r_Q$ . For any  $\lambda > 1$ , we denote by  $\lambda Q := B(c(Q), \lambda r_Q)$ . We write  $L^p$  for  $L^p(\mathbb{R}^n, \mathbb{R})$  or  $L^p(\mathbb{R}^n, \mathbb{C})$ . For a subset  $E \subset \mathbb{R}^n$  of finite and non-vanishing measure and  $f$  a locally integrable function, the average of  $f$  on  $E$  is defined by

$$\oint_E f dx := \frac{1}{|E|} \int_E f(x) dx.$$

Let us denote by  $\mathcal{Q}$  the collection of all balls in  $\mathbb{R}^n$ . We write  $\mathcal{M}$  for the maximal Hardy-Littlewood function:

$$\mathcal{M}f(x) = \sup_{\substack{Q \in \mathcal{Q} \\ x \in Q}} \oint_Q |f| dx.$$

For  $p \in (1, \infty)$ , we set  $\mathcal{M}_p f(x) = \mathcal{M}(|f|^p)(x)^{1/p}$ . The Fourier transform will be denoted by  $\mathcal{F}$  as an operator and we make use of the other usual notation  $\mathcal{F}(f) = \widehat{f}$  too.

In the current work, we aim to develop a multi-frequency analysis, based on the following lemma:

**Lemma 1.1** ([2]). *Let  $\Theta \subset \mathbb{R}^n$  be a finite collection of frequencies and  $Q$  be a ball. For every function  $\phi$  belonging to the subspace of  $L^2(3Q)$ , spanned by  $(e^{i\xi})_{\xi \in \Theta}$ , we have for  $p \in [1, 2]$*

$$(1) \quad \|\phi\|_{L^\infty(Q)} \lesssim (\#\Theta)^{\frac{1}{p}} \left( \oint_{3Q} |\phi|^p dx \right)^{\frac{1}{p}}.$$

*Remark 1.1.* In [2], this lemma is stated and proved in a one-dimensional setting. However, the proof only relies on the additive group structure of the ambient space by using translation operators. So the exact same proof can be extended to a multi-dimensional setting.

*Remark 1.2.* The question of extending the previous lemma for  $p \in (2, \infty)$  is still open in such a general situation. Of course, (1) is true for  $p = \infty$  and so it would be reasonable to expect the result for intermediate exponents  $p \in (2, \infty)$ . Unfortunately, the well-known interpolation theory does not apply here.

However, in some specific situations, we may extend this lemma for  $p \geq 2$ . Indeed, if  $p = 2k$  is an even integer then applying (1) with  $p = 2$  and  $\Theta^k := \{\theta_{i_1} + \dots + \theta_{i_k}, \theta_i \in \Theta\}$  to  $\phi^k$  yields

$$\begin{aligned} \|\phi\|_{L^\infty(Q)} &\lesssim \|\phi^k\|_{L^\infty(Q)}^{\frac{1}{k}} \\ &\lesssim (\#\Theta^k)^{\frac{1}{2k}} \left( \oint_{3Q} |\phi|^{2k} dx \right)^{\frac{1}{2k}} \\ &\simeq (\#\Theta^k)^{\frac{1}{p}} \left( \oint_{3Q} |\phi|^p dx \right)^{\frac{1}{p}}. \end{aligned}$$

By this way, we see that an extension of (1) for  $p \geq 2$  may be related to sharp combinatorial arguments, to estimate  $\#\Theta^k$  (a trivial bound is  $\#\Theta^k \leq (\#\Theta)^k$  which does not improve (1)).

We aim to obtain weighted estimates, involving Muckenhoupt's weights.

*Definition 1.1.* A weight  $\omega$  is a non-negative locally integrable function. We say that a weight  $\omega \in \mathbb{A}_p$ ,  $1 < p < \infty$ , if there exists a positive constant  $C$  such that for every ball  $Q$ ,

$$\left( \int_Q \omega \, dx \right) \left( \int_Q \omega^{1-p'} \, dx \right)^{p-1} \leq C.$$

For  $p = 1$ , we say that  $\omega \in \mathbb{A}_1$  if there is a positive constant  $C$  such that for every ball  $Q$ ,

$$\int_Q \omega \, dx \leq C \omega(y), \quad \text{for a.e. } y \in Q.$$

We write  $\mathbb{A}_\infty = \cup_{p \geq 1} \mathbb{A}_p$ .

We just recall that for  $p \in (1, \infty)$ , the maximal function  $\mathcal{M}$  is bounded on  $L^p(\omega)$  if and only if  $\omega \in \mathbb{A}_p$ . We also need to introduce the reverse Hölder classes.

*Definition 1.2.* A weight  $\omega \in RH_p$ ,  $1 < p < \infty$ , if there is a constant  $C$  such that for every ball  $Q$ ,

$$\left( \int_Q \omega^p \, dx \right)^{1/p} \leq C \left( \int_Q \omega \, dx \right).$$

It is well known that  $\mathbb{A}_\infty = \cup_{r > 1} RH_r$ . Thus, for  $p = 1$  it is understood that  $RH_1 = \mathbb{A}_\infty$ .

**1.1. Examples of multi-frequency Calderón-Zygmund operators.** Let us detail particular situations where such multi-frequency operators appear.

*The multi-frequency Hilbert transform.* As explained in the introduction, an example of such multi-frequency operators in the 1-dimensional setting is the multi-frequency Hilbert transform. In  $\mathbb{R}$ , consider an arbitrary collection of frequencies  $\Theta := (\xi_1, \dots, \xi_N)$  (assumed to be indexed by the increasing order  $\xi_1 < \xi_2 < \dots < \xi_N$ ). The associated multi-frequency Hilbert transform is the Fourier multiplier corresponding to the symbol

$$m(\xi) = \begin{cases} -1, & \xi < \xi_1 \\ (-1)^{j+1}, & \xi_j < \xi < \xi_{j+1} \\ (-1)^{N+1}, & \xi > \xi_N. \end{cases}$$

Associated to  $\Theta$ , we have a collection of disjoint intervals  $\Delta := \{(-\infty, \xi_1), (\xi_1, \xi_2), \dots, (\xi_N, \infty)\}$ . It is well-known by Rubio de Francia's work [13] that for  $q \in (1, 2]$ , the functional

$$(2) \quad f \rightarrow \left( \sum_{\omega \in \Delta} |\mathcal{F}^{-1}[\mathbf{1}_\omega \mathcal{F} f]|^q \right)^{\frac{1}{q}}$$

is bounded on  $L^p$  for  $p \in (q', \infty)$ .

The boundedness of the multi-frequency Hilbert transform is closely related to the understanding of (2) for  $q \rightarrow 1$ .

We point out that in Rubio de Francia's result, the obtained estimates do not depend on the collection of intervals  $\Delta$ . More precisely, excepted the end-point  $p = q'$ , the range  $(q', \infty)$  is optimal for a uniform (with respect to the collection  $\Delta$ )  $L^p$ -boundedness of (2). So it is natural that for  $q \rightarrow 1$  things are more difficult, which is illustrated by our multi-frequency Calderón-Zygmund analysis. Indeed, for example if one considers the particular case  $\Theta := (1, \dots, N)$ , then following the notations of Remark 1.2, we have  $\Theta^k = \{k, \dots, kN\}$  and so  $\#\Theta^k = k(N-1)+1 \simeq kN$ .

Hence, in this situation we have observed (see Remark 1.2) that we can extend Lemma 1.1 to exponents  $p \in [1, \infty]$  (the implicit constant appearing in (1) is only depending on  $p$ ). By this way, Theorem 0.2 can be improved and we obtain a better exponent

$$\gamma = \frac{tp}{s^2} + \left| \frac{1}{2} - \frac{1}{s} \right|.$$

Consequently, it seems that for the  $L^p$ -boundedness of the multi-frequency Hilbert transform, the collection  $\Theta$  could play an important role (which was not the case for the  $\ell^q$ -functional (2) with  $q' < p$ ).

*Multi-frequency operators coming from a covering of the frequency space.* Let  $(Q_j)_{j=1, \dots, N}$  be a family of disjoint cubes and  $\phi_j$  a smooth function with  $\widehat{\phi_j}$  supported and adapted to  $Q_j$ . Then consider the linear operator given by

$$T(f) = \sum_{j=1}^N \phi_j * f.$$

It is easy to check that  $T$  is a multi-frequency Calderón-Zygmund operator, associated to the collection  $\Theta := (\xi_1, \dots, \xi_N)$  where for every  $j$ ,  $\xi_j := c(Q_j)$  is the center of the ball  $Q_j$ . With  $r_j$  the radius of  $Q_j$ , we have the regularity estimate

$$\sum_{j=1}^N \left| \nabla_{(x,y)} e^{i\xi_j \cdot (x-y)} \phi_j(x-y) \right| \lesssim |x-y|^{-n-1} \sum_{j=1}^N \frac{(r_j |x-y|)^{n+1}}{(1+r_j |x-y|)^M},$$

for every integer  $M > 0$ .

So boundedness of  $T$  (Theorem 0.1) yields the inequality

$$(3) \quad \left\| \sum_{j=1}^N \phi_j * f \right\|_{L^p} \lesssim C(r_1, \dots, r_N) N^{\left| \frac{1}{p} - \frac{1}{2} \right|} \|f\|_{L^p},$$

with

$$C(r_1, \dots, r_N) := \sup_{t>0} \sum_{j=1}^N \frac{(r_j t)^{n+1}}{(1+r_j t)^M}.$$

Let us examine some particular situations:

- If the cubes  $(Q_j)_j$  have an equal side-length, then as for Proposition 4.1, simple arguments imply (3) for  $p \in [1, \infty]$  without the constant  $C(r_1, \dots, r_N)$ .
- If the collection  $(Q_j)_j$  is dyadic: it exists a point  $\xi_0$ ,  $d(Q_j, \xi_0) \simeq r_{Q_j} \simeq 2^j$  then Littlewood-Paley theory implies (3) without the factor  $N^{\left| \frac{1}{p} - \frac{1}{2} \right|}$  (in this case  $C(r_1, \dots, r_N) \simeq 1$ ).
- If the cubes  $(Q_j)_j$  have only the dyadic scale:  $r_{Q_j} \simeq 2^j$  (but no assumptions on the centers of the balls) then Littlewood-Paley theory cannot be used. However, our previous results can be applied in this situation and so (3) holds and  $C(r_1, \dots, r_N) \simeq 1$ .

We aim to use the new multi-frequency Calderón-Zygmund analysis to extend these inequalities with replacing the convolution operators by more general Calderón-Zygmund operators, still satisfying some orthogonality properties.

*Multi-frequency operators coming from variation norm estimates.* As explained in the introduction, the multi-frequency Calderón-Zygmund analysis has been first developed for proving a variation norm variant of a Bourgain's maximal inequality. So our results can be adapted in such a framework. For example, in [7] Grafakos, Martell and Soria have studied maximal inequalities of the form

$$\left\| \sup_{j=1,\dots,N} |T(e^{i\theta_j \cdot} f)| \right\|_{L^p} \lesssim \|f\|_{L^p}$$

where  $(\theta_j)_{j=1,\dots,N}$  is a collection of frequencies and  $T$  a fixed Calderón-Zygmund operator.

We can ask the same question, for a variation norm variant: for  $q \in [1, \infty)$  consider

$$\left( \sum_{j=1}^N |T(e^{i\theta_j \cdot} f)|^q \right)^{\frac{1}{q}}$$

and study its boundedness on  $L^p$ , with a sharp control of the behaviour with respect to  $N$ . By a linearization argument (involving Rademacher's functions), this  $\ell^q$ -functional can be realized as an average of modulated Calderón-Zygmund operators, associated to the collection  $\Theta := (\theta_j)_j$ .

## 2. UNWEIGHTED ESTIMATES FOR MULTI-FREQUENCY CALDERÓN-ZYGMUND OPERATORS

In this section, we aim to prove the weak  $L^1$ -estimate for a multi-frequency Calderón-Zygmund operator, then Theorem 0.1 will easily follow from interpolation and duality.

**Proposition 2.1.** *Let  $\Theta = (\xi_1, \dots, \xi_N)$  be a collection of  $N$  frequencies as above and  $T$  be a Calderón-Zygmund operator relatively to  $\Theta$ . Then  $T$  is of weak type  $(1, 1)$  with (uniformly with respect to  $N$ )*

$$\|T\|_{L^1 \rightarrow L^{1,\infty}} \lesssim N^{\frac{1}{2}}.$$

*Proof.* Consider  $f$  a function in  $L^1$  and  $\lambda > 0$ , we use the Calderón-Zygmund decomposition<sup>2</sup> of [10] related to the collection of frequencies  $\Theta$ . So the function  $f$  can be decomposed  $f = g + \sum_{J \in \mathbf{J}} b_J$  with the following properties:

- $\mathbf{J}$  is a collection of balls and  $(3J)_{J \in \mathbf{J}}$  has a bounded overlap;
- for each  $J \in \mathbf{J}$ ,  $b_J$  is supported in  $3J$ ;
- we have

$$(4) \quad \sum_{J \in \mathbf{J}} |J| \lesssim \sqrt{N} \|f\|_{L^1} \lambda^{-1};$$

- the “good part”  $g$  satisfies

$$(5) \quad \|g\|_{L^2}^2 \lesssim \|f\|_{L^1} \sqrt{N} \lambda;$$

- the cubes  $J$  satisfy

$$(6) \quad \|f\|_{L^1(J)} \lesssim |J| \lambda N^{-\frac{1}{2}}, \quad \|f - b_J\|_{L^2(J)} \lesssim \sqrt{|J|} \lambda;$$

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<sup>2</sup>In [10], the multi-frequency Calderón-Zygmund decomposition is only described in  $\mathbb{R}$ . The proof is a combination of Lemma 1.1 and the usual Calderón-Zygmund decomposition. Since both of them can be extended in a multi-dimensional framework, the multi-frequency Calderón-Zygmund decomposition performed in [10] still holds in  $\mathbb{R}^n$ .

• we have cancellation for all the frequencies of  $\Theta$ : for all  $j = 1, \dots, N$  and  $J \in \mathbf{J}$ ,  $\widehat{b}_J(\xi_j) = 0$ .  
We aim to estimate the measure of the level-set

$$\Upsilon_\lambda := \{x, |T(f)(x)| > \lambda\}.$$

With  $b = \sum_J b_J$ , we have

$$\begin{aligned} |\Upsilon_\lambda| &\leq |\{x, |T(g)(x)| > \lambda/2\}| + |\{x, |T(b)(x)| > \lambda/2\}| \\ &\lesssim \lambda^{-2} \|T(g)\|_{L^2}^2 + |\{x, |T(b)(x)| > \lambda/2\}| \\ &\lesssim \lambda^{-1} \sqrt{N} \|f\|_{L^1} + |\{x, |T(b)(x)| > \lambda/2\}|, \end{aligned}$$

where we used the  $L^2$ -boundedness of  $T$ . So it remains us to study the last term. Since (4), we get

$$\left| \bigcup_{J \in \mathbf{J}} 4J \right| \lesssim \sum_J |J| \lesssim \sqrt{N} \|f\|_{L^1} \lambda^{-1}.$$

Consequently, it only remains to estimate the measure of the set

$$O_\lambda := \left\{ x \in \left( \bigcup_{J \in \mathbf{J}} 4J \right)^c, \quad |T(b)(x)| > \lambda/2 \right\}.$$

Since

$$(7) \quad |O_\lambda| \lesssim \lambda^{-1} \sum_J \|T(b_J)\|_{L^1((2J)^c)},$$

it is sufficient to estimate the  $L^1$ -norms. Consider  $K$  the kernel of  $T$  and a point  $x_0 \in \left( \bigcup_{J \in \mathbf{J}} 4J \right)^c$ . Then, we can use the integral representation and we have

$$T(b)(x_0) = \int K(x_0, y) b(y) dy = \sum_J \int_{3J} K(x_0, y) b_J(y) dy.$$

To each  $J$ , we aim to take advantage of the cancellation properties of  $b_J$ , so we subtract the projection of  $[y \rightarrow K(x_0, y)]$  on the space, spanned by  $(e^{iy \cdot \eta})_{\eta \in \Theta}$ . So we have

$$\begin{aligned} T(b)(x_0) &= \sum_J \sum_{j=1}^N \int_{3J} \left[ K_j(x_0, y) - e^{i\xi_j \cdot c(J)} K_j(x_0, c(J)) e^{-i\xi_j \cdot y} \right] b_J(y) dy \\ &= \sum_J \sum_{j=1}^N \int_{3J} \left[ \tilde{K}_j(x_0, y) - \tilde{K}_j(x_0, c(J)) \right] e^{i\xi_j \cdot (x_0 - y)} b_J(y) dy \end{aligned}$$

where  $c(J)$  is the center of  $J$  and  $\tilde{K}_j(x, y) := K_j(x, y) e^{-i\xi_j \cdot (x - y)}$ . We then write

$$T_j(b)(x_0) := \int \left[ \tilde{K}_j(x_0, y) - \tilde{K}_j(x_0, c(J)) \right] e^{i\xi_j \cdot (x_0 - y)} b(y) dy.$$

such that  $T(b) = \sum_j T_j(b)$ . Due to the regularity assumption on  $K$  (and so on  $\tilde{K}_j$ ), it comes for  $y \in J$  and  $x_0 \in (2J)^c$

$$(8) \quad \sum_{j=1}^N \left| \tilde{K}_j(x_0, y) - \tilde{K}_j(x_0, c(J)) \right| \lesssim \frac{r_J}{|x_0 - y|^{n+1}}.$$



So we have

$$\|T(b_J)\|_{L^1((2J)^c)} \lesssim \iint_{|x-y| \geq r_J} \frac{r_J}{|x-y|^{n+1}} |b_J(y)| dx dy \lesssim \|b_J\|_{L^1} \lesssim |J|\lambda.$$

Finally, we obtain with (7) that

$$|O_\lambda| \lesssim \sum_J |J| \lesssim \sqrt{N} \|f\|_{L^1} \lambda^{-1},$$

which concludes the proof.  $\square$

*Remark 2.1.* Following [10], the bound of order  $N^{\frac{1}{2}}$  is optimal for the multi-frequency decomposition and for the weak- $L^1$  estimate.

### 3. WEIGHTED ESTIMATES FOR MULTI-FREQUENCY CALDERÓN-ZYGMUND OPERATORS

Aiming to obtain weighted estimates on such multi-frequency operators (using *Good-lambda inequalities*), we also have to define a suitable maximal sharp function, associated to a collection of frequencies.

*Definition 3.1* (Maximal sharp function). Let  $\Theta$  be a collection of  $N$  frequencies and  $s \in [1, \infty)$ . Consider a ball  $Q$ , we denote by  $\mathbb{P}_{\Theta, Q}$  the projection operator (in the  $L^s$ -sense) on the subspace of  $L^s(3Q)$ , spanned by  $(\exp i\xi \cdot)_{\xi \in \Theta}$ . Let us specify this projection operator: consider  $E$  the finite dimensional sub-space of  $L^s(3Q)$ , spanned by  $(e^{i\xi \cdot})_{\xi \in \Theta}$  and equipped with the  $L^s(3Q)$ -norm. Since  $E$  is of finite dimension, then for every  $f \in L^s(Q)$  there exists  $v := \mathbb{P}_{\Theta, Q}(f) \in E$  such that

$$\|f - v\|_{L^s(3Q)} = \inf_{\phi \in E} \|f - \phi\|_{L^s(3Q)}.$$

This projection operator may depend on  $s$ , which is not important for our purpose so this is implicit in the notation and we forget it.

Since  $0 \in E$ , we obviously have

$$(9) \quad \|\mathbb{P}_{\Theta, Q}(f)\|_{L^s(3Q)} \leq 2\|f\|_{L^s(Q)}.$$

Then, we may define the maximal sharp function

$$\mathcal{M}_{s, \Theta}^\sharp(f)(x_0) := \sup_{x_0 \in Q} \left( \int_Q |f - \mathbb{P}_{\Theta, Q}(f \mathbf{1}_Q)|^s dx \right)^{\frac{1}{s}}.$$

Note that the usual sharp maximal function is the one obtained for  $\Theta := \{0\}$  and in this situation it is well-known that the maximal sharp function satisfies a so-called Fefferman-Stein inequality (see [6]). We first prove an equivalent property for this generalised maximal sharp function:

**Proposition 3.1.** *Let  $s \in (1, \infty)$ ,  $t \in [1, \infty)$  and  $p \in (s, \infty)$  be fixed. Then for every function  $f \in L^s$  and every weight  $\omega \in RH_{t'}$ , we have for every  $p \geq s$*

$$\|f\|_{L^p(\omega)} \lesssim N^{\frac{tp}{s} \max\{\frac{1}{2}, \frac{1}{s}\}} \left\| \mathcal{M}_{s, \Theta}^\sharp(f) \right\|_{L^p(\omega)}.$$

The proof relies on a *Good-lambda inequality* and Lemma 1.1.

*Proof.* We make use on the abstract theory developed in [1] by Auscher and Martell. We also follow notations of [1, Theorem 3.1]. Indeed, for each ball  $Q \subset \mathbb{R}^n$  we have the following

$$F(x) := |f(x)|^s \lesssim |f(x) - \mathbb{P}_{\Theta, Q}(f\mathbf{1}_Q)(x)|^s + |\mathbb{P}_{\Theta, Q}(f\mathbf{1}_Q)(x)|^s := G_Q(x) + H_Q(x).$$

By definition, it comes

$$\int_Q G_Q dx \leq \inf_Q \mathcal{M}_{s, \Theta}^\#(f)^s$$

and following Lemma 1.1 (with (9))

$$\begin{aligned} \sup_{x \in Q} H_Q &= \|\mathbb{P}_{\Theta, Q}(f\mathbf{1}_Q)\|_{L^\infty(Q)}^s \lesssim N^{s \max\{\frac{1}{2}, \frac{1}{s}\}} \left( \int_{3Q} |\mathbb{P}_{\Theta, Q}(f\mathbf{1}_Q)|^s dx \right) \\ &\lesssim N^{s \max\{\frac{1}{2}, \frac{1}{s}\}} \left( \int_Q |f|^s dx \right) \lesssim N^{s \max\{\frac{1}{2}, \frac{1}{s}\}} \inf_Q \mathcal{M}F. \end{aligned}$$

So we can apply [1, Theorem 3.1] (with  $q = \infty$  and  $a \simeq N^{s \max\{\frac{1}{2}, \frac{1}{s}\}}$ ) and by checking the behaviour of the constants with respect to “ $a$ ” in its proof, we obtain for every  $p \geq 1$

$$\|\mathcal{M}_s(f)^s\|_{L^p(\omega)} \lesssim N^{spt \max\{\frac{1}{2}, \frac{1}{s}\}} \left\| \mathcal{M}_{s, \Theta}^\#(f)^s \right\|_{L^p(\omega)},$$

which yields the desired result.  $\square$

Then, we evaluate a multi-frequency Calderón-Zygmund operator via this new maximal sharp function.

**Proposition 3.2.** *Let  $T$  be a Calderón-Zygmund operator relatively to  $\Theta$  and  $s \in (1, \infty)$ . Then, we have the following pointwise estimate:*

$$\mathcal{M}_{s, \Theta}^\#(T(f)) \lesssim N^{|\frac{1}{s} - \frac{1}{2}|} \mathcal{M}_s(f).$$

*Proof.* We follow the well-known proof for usual Calderón-Zygmund operators and adapt the arguments to the current situation. So consider a point  $x_0$  and a ball  $Q \subset \mathbb{R}^n$  containing  $x_0$ , we have to estimate

$$\left( \int_Q |T(f) - \mathbb{P}_{\Theta, Q}(T(f)\mathbf{1}_Q)|^s dx \right)^{\frac{1}{s}}.$$

We split the function into a local part  $f_0$  and an off-diagonal part  $f_\infty$ :

$$f = f_0 + f_\infty := f\mathbf{1}_{10Q} + f\mathbf{1}_{(10Q)^c}.$$

By definition of the projection operator, we know that

$$\begin{aligned} \left( \int_Q |T(f) - \mathbb{P}_{\Theta, Q}(T(f)\mathbf{1}_Q)|^s dx \right)^{\frac{1}{s}} &\leq \left( \int_Q |T(f) - \mathbb{P}_{\Theta, Q}(T(f_\infty)\mathbf{1}_Q)|^s dx \right)^{\frac{1}{s}} \\ &\leq \left( \int_Q |T(f_0)|^s dx \right)^{\frac{1}{s}} + \left( \int_Q |T(f_\infty) - \mathbb{P}_{\Theta, Q}(T(f_\infty)\mathbf{1}_Q)|^s dx \right)^{\frac{1}{s}}. \end{aligned}$$

For the local part, we use boundedness in  $L^s$  of the operator  $T$  (Proposition 2.1), hence

$$\begin{aligned} \left( \int_Q |T(f_0)|^s dx \right)^{\frac{1}{s}} &\lesssim |Q|^{-\frac{1}{s}} \|T(f_0)\|_{L^s(Q)} \lesssim N^{\left(\frac{1}{2}-\frac{1}{s}\right)} \left( |Q|^{-\frac{1}{s}} \|f_0\|_{L^s} \right) \\ &\lesssim N^{\left|\frac{1}{2}-\frac{1}{s}\right|} \mathcal{M}_s(f)(x_0). \end{aligned}$$

Then let us focus on the second part, involving  $f_\infty$ .

We use the decomposition (with an integral representation) since we are in the off-diagonal case: for  $x \in Q$

$$T(f_\infty)(x) = \sum_{j=1}^N \int K_j(x, y) f_\infty(y) dy.$$

Consider the following function, defined on  $3Q$  by (where  $c(Q)$  is the center of  $Q$ )

$$\Phi := x \in 3Q \rightarrow \sum_{j=1}^N \int e^{i\xi_j \cdot (x - c(Q))} K_j(c(Q), y) f_\infty(y) dy.$$

So  $\Phi \in E$  (see Definition 3.1) and hence

$$(10) \quad \left( \int_Q |T(f_\infty) - \mathbb{P}_{\Theta, Q}(T(f_\infty) \mathbf{1}_Q)|^s dx \right)^{\frac{1}{s}} \leq \left( \int_Q |T(f_\infty) - \Phi|^s dx \right)^{\frac{1}{s}}.$$

If we set  $\tilde{K}_j(x, z) := K_j(x, z) e^{-i\xi_j \cdot (x - z)}$ , then

$$T(f_\infty)(x) - \Phi(x) = \sum_j \int \left[ \tilde{K}_j(x, y) - \tilde{K}_j(c(Q), y) \right] e^{i\xi_j \cdot (x - y)} f_\infty(y) dy.$$

From the regularity assumption on the kernels  $K_j$ 's, we have for  $y \in (10Q)^c$

$$(11) \quad \sum_j \left| \tilde{K}_j(x, y) - \tilde{K}_j(c(Q), y) \right| \lesssim r_Q \sup_{z \in Q} \sum_j \left| \nabla_x \tilde{K}_j(z, y) \right| \lesssim r_Q^{-n} \left( 1 + \frac{d(y, Q)}{r_Q} \right)^{-n-1}.$$

We also have (since  $y \in (10Q)^c$  and  $x, c(Q) \in Q$ )

$$\begin{aligned} |T(f_\infty)(x) - \Phi(x)| &\lesssim \int_{|z| \geq 10r_Q} r_Q^{-n} \left( 1 + \frac{|x - c(Q) - z|}{r_Q} \right)^{-n-1} |f(c(Q) + z)| dz \\ &\lesssim \int_{|z| \geq 5r_Q} r_Q^{-n} \left( 1 + \frac{|z|}{r_Q} \right)^{-n-1} |f(x_0 + z)| dz \\ &\lesssim \mathcal{M}(f)(x_0), \end{aligned}$$

which concludes the proof.  $\square$

We obtain the following corollary:

**Corollary 3.3.** *Let  $\Theta$  be a collection of  $N$  frequencies. For  $p \in (2, \infty)$ ,  $s \in [2, p)$  and  $t \in (1, \infty)$ , a multi-frequency Calderón-Zygmund operator  $T$  is bounded on  $L^p(\omega)$  for every weight  $\omega \in RH_{t'} \cap \mathbb{A}_{\frac{p}{s}}^{\pm}$  with*

$$\|T\|_{L^p(\omega) \rightarrow L^p(\omega)} \lesssim N^{\frac{tp}{2s} + \left(\frac{1}{2} - \frac{1}{s}\right)}.$$

*Proof.* Using Propositions 3.1 and 3.2, it follows that for  $p > s \geq 2$  (assuming  $\omega \in \mathbb{A}_{\frac{p}{s}}$ )

$$\begin{aligned} \|T(f)\|_{L^p(\omega)} &\lesssim N^{\frac{tp}{2s}} \left\| \mathcal{M}_{s,\Theta}^\sharp[T(f)] \right\|_{L^p(\omega)} \\ &\lesssim N^{\frac{tp}{2s} + (\frac{1}{2} - \frac{1}{s})} \|\mathcal{M}_s(f)\|_{L^p(\omega)} \\ &\lesssim N^{\frac{tp}{2s} + (\frac{1}{2} - \frac{1}{s})} \|f\|_{L^p(\omega)}, \end{aligned}$$

where we used weighted boundedness of the maximal function since  $\omega \in \mathbb{A}_{\frac{p}{s}}$ .  $\square$

As explained in the introduction, this estimate is only interesting when the exponent  $\frac{tp}{2s} + (\frac{1}{2} - \frac{1}{s})$  is lower than 1.

#### 4. CONNEXION TO BOCHNER-RIESZ MULTIPLIERS

In this section, we aim to describe how such arguments could be applied to generalized Bochner-Riesz multipliers. Weighted estimates for Bochner-Riesz multipliers has been initiated in [15, 5, 4]. We first emphasize that we do not pretend to obtain new weighted estimates for Bochner-Riesz multipliers. But we only want to describe here a new point of view and a new approach for such estimates, which will be the subject of a future investigation. Such an application is a great motivation for pursuing the study of a multi-frequency Calderón-Zygmund analysis.

Consider also  $\Omega$  a bounded open subset of  $\mathbb{R}^n$  such that its boundary  $\Gamma := \overline{\Omega} \setminus \Omega$  is an hyper-manifold of Hausdorff dimension  $n - 1$ . For  $\delta > 0$ , we then define the generalized Bochner-Riesz multiplier, given by

$$R_{\Omega,\delta}(f)(x) := \int_{\Omega} e^{ix \cdot \xi} \widehat{f}(\xi) m_{\delta} d\xi,$$

where  $m_{\delta}$  is a smooth symbol supported in  $\overline{\Omega}$  and satisfying in  $\Omega$

$$|\partial^{\alpha} m_{\delta}(\xi)| \lesssim d(\xi, \Gamma)^{\delta - |\alpha|}.$$

We first use a Whitney covering  $(O_i)_i$  of  $\Omega$ . That is a collection of sub-balls such that

- the collection  $(O_i)_i$  covers  $\Omega$  and has a bounded overlap;
- the radius  $r_{O_i}$  is equivalent to  $d(O_i, \Gamma)$ .

Associated to this collection, we build a partition of the unity  $(\chi_i)_i$  of smooth functions such that  $\chi_i$  is supported on  $O_i$  with

$$\sum_i \chi_i(\xi) = \mathbf{1}_{\Omega}(\xi)$$

and  $\|\partial^{\alpha} \chi_i\|_{\infty} \lesssim r_{O_i}^{-|\alpha|}$ .

Then,  $R_{\delta}$  may be written as

$$R_{\delta}(f)(x) = \sum_{j=-\infty}^{\infty} T_j(f)(x),$$

with

$$\begin{aligned} T_j(f)(x) &:= \sum_{\substack{l_i \\ 2^j \leq r_{O_l} < 2^{j+1}}} \int_{\Omega} e^{ix \cdot \xi} \widehat{f}(\xi) m_{\delta}(\xi) \chi_l(\xi) d\xi \\ (12) \quad &= 2^{j\delta} U_j(f)(x), \end{aligned}$$

where we set

$$U_j(f)(x) := \sum_{\substack{l_i \\ 2^j \leq r_{O_l} < 2^{j+1}}} \int_{\Omega} e^{ix \cdot \xi} \widehat{f}(\xi) (2^{-j\delta} m_{\delta}(\xi)) \chi_l(\xi) d\xi.$$

**Observation :** The main idea is to observe that the operator  $U_j$  is a multi-frequency Calderón-Zygmund operator associated to the collection

$$\Theta_j := \{c(O_l), 2^j \leq r_{O_l} < 2^{j+1}\} \quad \text{with} \quad \sharp \Theta_j \simeq 2^{-j(n-1)}.$$

However, these operators have specific properties, one of them is that the considered balls have equivalent radius, which means that these operators have only one scale  $2^j$ . For example, this observation allows us to easily prove some boundedness:

**Proposition 4.1.** *Uniformly with  $j \lesssim 0$ , the multiplier  $U_j$  is a convolution operation with a kernel  $K_j$  satisfying*

$$\|K_j\|_{L^1} \lesssim 2^{-j \frac{n-1}{2}}.$$

Hence, it follows that  $U_j$  is bounded on Lebesgue space  $L^p$  for every  $p \in [1, \infty]$ . Moreover for every  $s \in [1, 2]$ ,  $p \in (s, \infty)$  and every weight  $\omega \in \mathbb{A}_{\frac{p}{s}}$ ,  $U_j$  is bounded on  $L^p(\omega)$  with

$$\|U_j\|_{L^p(\omega) \rightarrow L^p(\omega)} \lesssim 2^{-j \frac{n-1}{s}}.$$

*Proof.* The operator  $U_j$  is a Fourier multiplier, associated to the symbol

$$\sigma_j(\xi) := \sum_{\substack{l_i \\ 2^j \leq r_{O_l} < 2^{j+1}}} (2^{-j\delta} m_{\delta}(\xi)) \chi_l(\xi).$$

Since the considered balls  $(O_l)_l$  are almost disjoint, it comes that

$$\|\sigma_j\|_{L^2} \lesssim |\{\xi, d(\xi, \partial\Omega) \simeq 2^j\}|^{\frac{1}{2}} \lesssim 2^{\frac{j}{2}}.$$

Moreover, using regularity assumptions on  $m_{\delta}$ , we deduce that for every  $\alpha$

$$\|\partial^{\alpha} \sigma_j\|_{L^2} \lesssim 2^{-j|\alpha|} |\{\xi, d(\xi, \partial\Omega) \simeq 2^j\}|^{\frac{1}{2}} \lesssim 2^{j(\frac{1}{2}-|\alpha|)}.$$

So with  $K_j := \mathcal{F}(\sigma_j)$ , it follows that for any integer  $M$

$$(13) \quad \|(1 + 2^j |\cdot|)^M K_j\|_{L^2} \lesssim 2^{\frac{j}{2}}.$$

Hence

$$\|K_j\|_{L^1} \lesssim 2^{-j \frac{n-1}{2}}.$$

Using Minkowski inequality, we deduce that for every  $p \in [1, \infty]$

$$\|U_j\|_{L^p \rightarrow L^p} \lesssim \|K_j\|_{L^1} \lesssim 2^{-j \frac{n-1}{2}}.$$

Let us now focus on the second claim about weighted estimates. Using integrations by parts for computing the kernel  $K_j$ , it comes for any integer  $M$

$$(14) \quad \|(1 + 2^j |\cdot|)^M K_j\|_{L^\infty} \lesssim 2^j.$$

By interpolation with (13), for  $s \in [1, 2]$  we get

$$(15) \quad \|(1 + 2^j |\cdot|)^M K_j\|_{L^{s'}} \lesssim 2^{\frac{j}{s}},$$

which gives

$$U_j(f) \lesssim 2^{-j \frac{n-1}{s}} \mathcal{M}_s(f).$$

Hence, for every  $p > s$  and every weight  $\omega \in \mathbb{A}_{\frac{p}{s}}$

$$\|U_j\|_{L^p(\omega) \rightarrow L^p(\omega)} \lesssim 2^{-j \frac{n-1}{s}}.$$

□

In this context,  $\sharp\Theta_j \simeq 2^{-j(n-1)}$ , so the constant  $2^{-j \frac{n-1}{s}}$  is equivalent to  $(\sharp\Theta_j)^{\frac{1}{s}}$  and this is a better constant than the one obtained in Corollary 3.3 (for a subclass of  $\mathbb{A}_{\frac{p}{s}}$  weights).

So improving these “easy bounds” means to obtain inequalities such as

$$\|U_j\|_{L^p(\omega) \rightarrow L^p(\omega)} \lesssim (\sharp\Theta_j)^\gamma$$

for some better exponent  $\gamma < \frac{1}{s}$ .

Let us finish by suggesting how could we get improvements of our approach to get interesting results for Bochner-Riesz multipliers:

**Question :** The general approach, developed in the previous section, only allows to get an exponent

$$\gamma = \frac{tp}{2s} + \left( \frac{1}{2} - \frac{1}{s} \right)$$

(with some  $s \in [2, p)$ ) which is bigger than  $\frac{1}{2}$  (since  $p > s \geq 2$  and  $t > 1$ ). So to improve this exponent  $\gamma$ , two things seem to be crucial:

- to extend the use of Lemma 1.1 for  $p \geq 2$  which would allow us to get an exponent  $\frac{tp}{s^2}$  instead of  $\frac{tp}{2s}$ ;
- to use the geometry of the boundary  $\Gamma$  to get better exponents, even for the unweighted estimates. Indeed, for example for the unit ball (using its non vanishing curvature), we know that (see [9, 14])

$$\|U_j\|_{L^p \rightarrow L^p} \lesssim 2^{-j\delta(p)}$$

with if  $n = 2$

$$\delta(p) := \max \left\{ 2 \left| \frac{1}{2} - \frac{1}{p} \right| - \frac{1}{2}, 0 \right\}.$$

and if  $n \geq 3$  and  $p \geq \frac{2(n+2)}{n}$  or  $p \leq \frac{2(n+2)}{n+4}$

$$\delta(p) := \max \left\{ n \left| \frac{1}{2} - \frac{1}{p} \right| - \frac{1}{2}, 0 \right\}.$$

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